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A 1-Local Asymptotic 13/9-Competitive Algorithm for Multicoloring Hexagonal Graphs

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Abstract In the frequency allocation problem, we are given a mobile telephone network, whose geographical coverage area is divided into cells, wherein phone calls are serviced by assigning frequencies to them so that no two calls emanating from the same or neighboring cells are assigned the same frequency. The problem is to use the frequencies efficiently, i.e., minimize the span of frequencies used. The frequency allocation problem can be regarded as a multicoloring problem on a weighted hexagonal graph. In this paper, we give a 1-local asymptotic 4/3-competitive distributed algorithm for multicoloring a triangle-free hexagonal graph, which is a special case of hexagonal graph. Based on this result, we then propose a 1-local asymptotic 13/9-competitive algorithm for multicoloring the (general-case) hexagonal graph, thereby improving the previous 1-local 3/2-competitive algorithm.

Keywords Online algorithm · Multicoloring · Hexagonal graphs

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1 Introduction

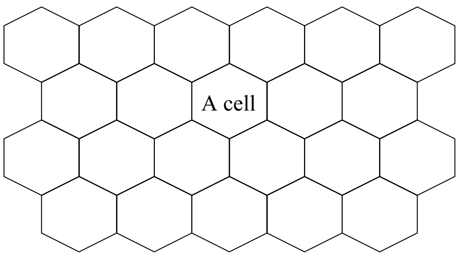
Wireless communication based on Frequency Division Multiplexing (FDM) technology is widely used in the area of mobile computing today. In such FDM networks, a geographic area is divided into small cellular regions or *cells*, each containing one base station. Base stations communicate with each other via a high-speed wired network. Calls between any two clients (even within the same cell) must be established through base stations. When a call arrives, the nearest base station must allocate a frequency from the available spectrum to the call without causing any interference to other calls. In practice, when the same frequency is assigned to two different calls emanating from cells that are geographically close to each other, interference may occur which distorts the radio signals. To avoid interference, the temptation is to use many frequencies. However, spectrum is a scarce resource, so efficient utilization of the available spectrum is essential for FDM networks.

The *frequency allocation problem*, including the off-line version and online version, has been extensively studied [1–4, 6, 9, 10, 12, 13, 15]. The research on this problem is mainly focused on the cellular network, where cells are hexagonal regions as shown in Fig. 1. In the off-line problem (the calls to be serviced are known *a priori*), McDiarmid and Reed [12] have shown that minimizing the span of frequencies to satisfy all the call requests is NP-hard, they also proved that 4/3 is the lower bound of approximation ratio. Further, two 4/3-approximation algorithms were given in [12] and [14] respectively.

For the online version, there are mainly three strategies, which have been well studied: the *fixed allocation assignment* (FAA) [11], the *greedy algorithm* (Greedy) [5] and the *hybrid algorithm* [4]. FAA partitions cells into independent sets which are each assigned a separate set of frequencies. It is easy to see that FAA is 3-competitive as cellular networks are 3-colorable [5]. Greedy assigns the minimum available number (frequency) to a new call so that the call does not interfere with calls of the same or adjacent cells. Caragiannis et al. [5] proved that the competitive ratio of Greedy in cellular network is at least 17/7 and at most 2.5. Chan et al. [3] closed the gap and proved that Greedy is 17/7-competitive. Furthermore, Chan et al. [4] gave a 2-competitive algorithm, say *hybrid*, which can be regarded as a combination of FAA and Greedy. Since the lower bound of competitive ratio of frequency allocation in cellular networks is also 2, *hybrid* is optimal.

The frequency allocation problem in cellular network can be abstracted to the problem of *multicoloring a weighted hexagonal graph*, in which each vertex has a positive weight which specifies how many different colors have to be assigned to the

Fig. 1 Example of a cellular network (with hexagonal cells)



vertex. Given the constraint that the same color cannot be assigned to the same or adjacent vertices, the target is to minimize the number of assigned colors.

In frequency allocation problems, the size of cellular network is very large, when handling a call request, the computation will be very complex if all information of the whole network is needed. In reality, each server in a cell only knows its position before processing the request sequence of calls; when satisfying call requests, each server only knows its local information, i.e., some information within a fixed distance. Such kind of processing gives us a motivation to study distributed algorithms. In this paper, we focus on *distributed algorithms* for the multicoloring, i.e., each vertex is an independent server, which runs the algorithm to assign multicolors to the vertex based on what is known as k-*local* information. The concept of k-*local* distributed algorithms was introduced by Janssen et al. [8], where an algorithm is k-local if the computation at a vertex depends only on the information of the neighboring vertices of at most k distance away (suppose each edge has unit distance). Similar to frequency allocation problem, we can assume that in multicoloring problem, each vertex also knows its position in the graph.

In [8], Janssen et al. proved (the next lemma) that a k-local c-approximate off-line algorithm can be easily converted to a k-local c-competitive online algorithm. Thus, to design a k-local online algorithm, we need only to focus on the k-local off-line problem.

Lemma 1 [8] *Let A be a k-local c-approximate off-line algorithm for multicoloring. Then A can be converted into a k-local c-competitive online algorithm for multicoloring.*

The problem of multicoloring a hexagonal graph is hard. But for some various graph classes, this problem may have a better performance. An interesting induced graph, *triangle-free hexagonal graph*, has been studied for the multicoloring problem [7, 16]. A graph is *triangle-free* if there are no 3-cliques in the graph, i.e., there are no three mutually-adjacent vertices with positive weights. An example of a triangle-free hexagonal graph is shown in Fig. 2.

The best known competitive ratios for 0-, 1-, 2- and 4-local distributed algorithms for multicoloring on (general) hexagonal graphs are 3, 3/2, 4/3 and 4/3, respectively [8, 17]. It is possible to do better for triangle-free hexagonal graphs. For

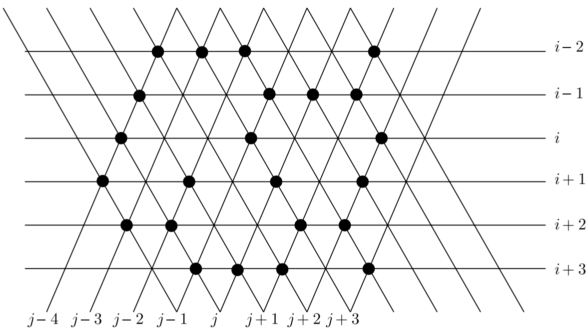


Fig. 2 An example of a triangle-free hexagonal graph, where *solid circles* are vertices with positive weights

example, in [16], a 2-local 5/4 competitive algorithm was given, and an inductive proof for the 7/6 ratio was reported in [7].

The remaining of this paper is organized as follows. In Sect. 2, we introduce some preliminary terminology to be used in this paper. In Sect. 3, we give a 1-local asymptotic 4/3-competitive algorithm for multicoloring a triangle-free hexagonal graph. Based on this result, we then propose, in Sect. 4, a 1-local asymptotic 13/9-competitive algorithm for the multicoloring problem in hexagonal graph, which improves the previous 3/2-competitive result. Finally, we give the conclusion in Sect. 5.

2 Preliminary Terminology

Given a hexagonal graph with a non-negative weight at each vertex, we use a 2-coordinate system to represent each vertex. In particular, referring to the lines shown in Fig. 2, each vertex can be represented by coordinate (i, j) where i is the coordinate for the horizontal line, j for the up-sloping line. For example, a vertex with coordinate (i, j) and its six neighboring vertices, denoted by UL, L, DL, UR, R and DR, are represented as shown in Fig. 3.

Next, we define the *parity* of a vertex with respect to its various neighbors. We say that the parity of a vertex v with coordinate (i, j) is:

1. *odd* (alternatively, *even*) with respect to its L or R neighbor if $j \equiv 1 \mod 2$ (correspondingly, $j \equiv 0 \mod 2$);
2. *odd* (alternatively, *even*) with respect to its UL or DR neighbor if $i \equiv 1 \mod 2$ (correspondingly, $i \equiv 0 \mod 2$);
3. *odd* (alternatively, *even*) with respect to its DL or UR neighbor if $i \equiv 1 \mod 2$ (correspondingly, $i \equiv 0 \mod 2$).

Let w_v be the weight of vertex v, which corresponds to the number of colors needed to multicolor v. After the multicoloring assignment, each vertex v will be assigned a *set* F_v *of colors*, such that $F_v \subset Z^+$, a set of positive integers, and $|F_v| = w_v$, where, for any two adjacent vertices u and v, $F_u \cap F_v = \phi$.

In order to help with this multicoloring assignment, we shall partition the set of vertices into three sets, each associated with a base color which denotes a separate set of colors (integers). Since a hexagonal graph is 3-colorable, we use three *base colors*,

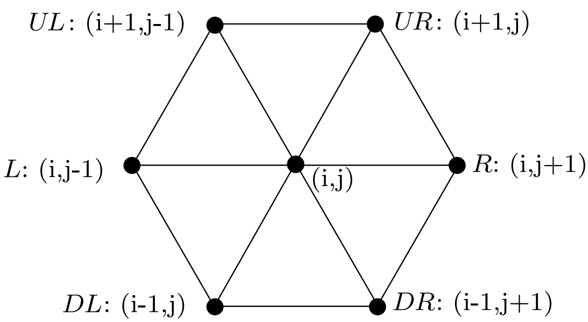


Fig. 3 Coordinates of a vertex (i, j) and its neighboring vertices

say Red, Green and Blue, coloring all the vertices of a hexagonal graph, such that each vertex colored with one of the three *base colors*, and no two adjacent vertices are of the same *base color* and similarly the same color. Furthermore, we assume a transitive order on these three base colors: namely, Red < Green < Blue.

3 Multicoloring in Triangle-Free Hexagonal Graphs

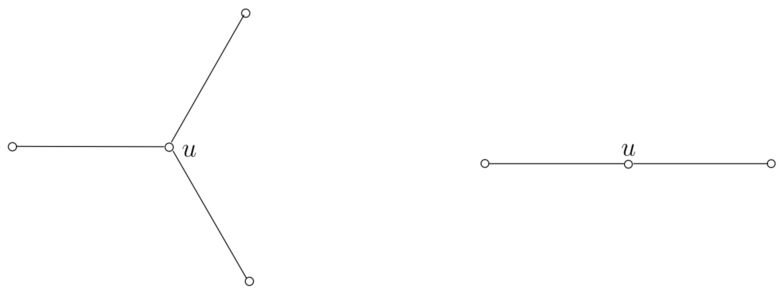
In this section, we shall study the problem of multicoloring a special type of hexagonal graph, say *triangle-free hexagonal graph*. In the next section, we will show that finding a good solution for this problem will lead to an algorithm for finding a good solution for a general hexagonal graph.

A graph is *triangle-free* if no three mutually-adjacent vertices have positive weights. For a given vertex u with positive weight w_u, from this definition of triangle-free graph, only two possible configurations may exist for the structure of neighbors with positive weights, which are shown in Fig. 4. It is easy to see that if u has 3 neighbors with positive weights, the neighboring vertices are of the same color. On the other hand, if the neighbors are of different colors, u has at most 2 neighbors with positive weights. There exists a simple structure in triangle-free graph, i.e., a vertex has only one neighbor, we can regard this structure as the case in Fig. 4(b).

Consider vertex u with positive weight w_u. Compute $c_u = w_u + \max\{w_v \mid v \text{ is } u\text{'s neighbor}\}$. c_u is the weight of the maximum 2-clique adjacent to u, which also gives the minimum number of colors needed for multicoloring a triangle-free hexagonal graph. From the definition of c_u, any feasible coloring of vertex u and its neighbors requires at least c_u colors.

Let $d_u = \lceil c_u/3 \rceil$. For each vertex u, define four color sets, each of size d_u:

1. $\text{colorset}_u(\text{Red}) = \{j \in \{1, \ldots, 4d_u\} \mid j = 1 \mod 4\}$,
2. $\text{colorset}_u(\text{Green}) = \{j \in \{1, \ldots, 4d_u\} \mid j = 2 \mod 4\}$,
3. $\text{colorset}_u(\text{Blue}) = \{j \in \{1, \ldots, 4d_u\} \mid j = 3 \mod 4\}$, and
4. $\text{extraset}_u = \{j \in \{1, \ldots, 4d_u\} \mid j = 0 \mod 4\}$.



(a) Structure A: neighbors with the same base color

(b) Structure B: neighbors with different base colors

Fig. 4 Structure of neighbors with positive weights

We will give a strategy to multicolor any vertex u with weight w_u by assigning w_u colors from the above four sets so that no adjacent vertices are assigned the same color. The assignment strategy assigns multicolors to u according to its base color and neighboring structure, which can be described as follows.

Assume vertex u with base color X has neighboring structure A, i.e. all its neighbors are of the same base color $Y \neq X$. Let the third base color be Z where $Z \neq X$ and $Z \neq Y$. In this case, our strategy assigns multicolors to vertex u first from $\text{colorset}_u(X)$, then $\text{colorset}_u(Z)$ if $\text{colorset}_u(X)$ is not large enough, and finally $\text{colorset}_u(Y)$ if both $\text{colorset}_u(X)$ and $\text{colorset}_u(Z)$ are still not large enough.

On the other hand, if vertex u with base color X has neighboring structure B, then all three base colors appear in u and its neighbors. The strategy first assigns multicolors from $\text{colorset}_u(X)$, and if not large enough, then from the extra color set extraset_u and finally from $\text{colorset}_u(Y)$ where base color $Y > Z \neq X$.

Note that in both cases, the colors in each color set may be assigned either from bottom to top or from top to bottom (i.e., from the lowest integer to the highest or from the highest to the lowest), depending on the base color or the parity of the vertex so as to avoid conflicts.

The Strategy

1. If vertex u has no neighbors, just assign w_u colors from 1 to w_u.
2. If vertex u with base color X has neighboring structure A (Fig. 4(a)), let Y be the base color of u's neighbors and Z be the other third color. Assign w_u multicolors to vertex u as follows:
   (a) Assign colors from $\text{colorset}_u(X)$ in bottom-to-top order.
   (b) If not enough, assign colors from $\text{colorset}_u(Z)$ in bottom-to-top order if $X < Y$; top-to-bottom otherwise.
   (c) If still not enough, assign colors from $\text{colorset}_u(Y)$ in top-to-bottom order.
3. If vertex u with base color X has neighboring structure B (Fig. 4(b)), let Y and Z be the base colors of the left neighbor and the right neighbor, respectively. Without loss of generality, assume $Y > Z$. Assign w_u multicolors to vertex u as follows:
   (a) Assign colors from $\text{colorset}_u(X)$ in bottom-to-top order.
   (b) If not enough, assign colors from extraset_u in bottom-to-top order if u is *odd* with respect to its two neighbors; top-to-bottom order otherwise.
   (c) If still not enough, assign colors from $\text{colorset}_u(Y)$ in top-to-bottom order.

Theorem 2 *The above strategy is* 1*-local and can solve the multicoloring problem in triangle-free hexagonal graphs with an asymptotic approximation ratio* 4/3.

Proof From the description of the strategy, it is clear that the colors assigned to any vertex depend only on neighboring information within distance 1, and thus, the strategy is 1-local.

To prove that the above strategy solves the multicoloring problem, we must prove that the colors assigned to any two adjacent vertices u and v are all different. As it turns out, we need to analyze the three cases shown in Fig. 5. X and Y denote the two respective different base colors of u and v. It is easy to see that different kinds of color sets of u and v have no common colors. For example, $\text{colorset}_u(\text{Red}) \cap \text{extraset}_v = \emptyset$.

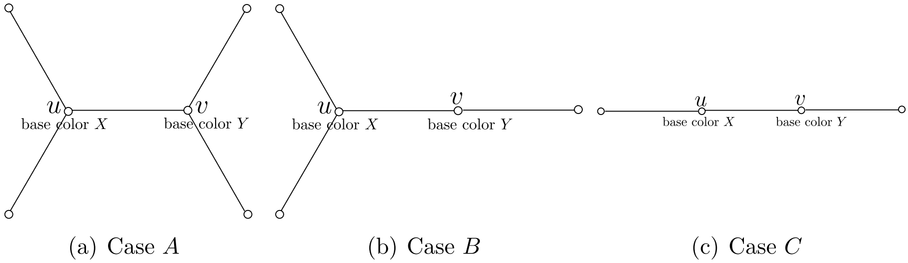


Fig. 5 The local structure of vertices u and v

From the definition of c_u, we have $c_u \geq w_u + w_v$ and, since $d_u = \lceil c_u/3 \rceil$, $c_u \leq 3d_u$.

For Case A, the strategy would assign colors to u from $\text{colorset}_u(X)$, then $\text{colorset}_u(Z)$ and then $\text{colorset}_u(Y)$, and would assign colors to v from $\text{colorset}_v(Y)$, then $\text{colorset}_v(Z)$ and then $\text{colorset}_v(X)$. If the assigned colors of u and v are from different color sets, there is no confliction. Otherwise, u and v use colors from the same color set, $\text{colorset}(X) = \text{colorset}_u(X) \cup \text{colorset}_v(X)$, $\text{colorset}(Y) = \text{colorset}_u(Y) \cup \text{colorset}_v(Y)$ or $\text{colorset}(Z) = \text{colorset}_u(Z) \cup \text{colorset}_v(Z)$. We study these three cases in the following:

(A-1) where u is assigned colors from $\text{colorset}_u(X)$ and v, after exhausting all colors in $\text{colorset}_v(Y)$ and $\text{colorset}_v(Z)$, is assigned colors from $\text{colorset}_v(X)$. This means the weight w_v is very large. Since $w_u + w_v \leq c_u \leq 3d_u$, $w_u + w_v \leq c_v \leq 3d_v$, and since u and v use $\text{colorset}(X) = \text{colorset}_u(X) \cup \text{colorset}_v(X)$ from opposite directions, u and v will not be assigned the same color;

(A-2) where u is assigned colors from $\text{colorset}_u(Z)$ and v is assigned colors from $\text{colorset}_v(Z)$. Then, all the colors in $\text{colorset}_u(X)$ are assigned to u and all the colors in $\text{colorset}_v(Y)$ are assigned to v. Since $w_u + w_v \leq \min\{3d_u, 3d_v\}$, and since u and v use $\text{colorset}(Z) = \text{colorset}_u(Z) \cup \text{colorset}_v(Z)$ from opposite directions (depending on whether $X < Y$ or otherwise), u and v will not be assigned the same color;

(A-3) where u is assigned colors from $\text{colorset}_u(Y)$ and v is assigned colors from $\text{colorset}_v(Y)$. By similar analysis as in case (A-1), we can say that u and v will not be assigned the same color.

In Case B, the strategy would assign colors to u from $\text{colorset}_u(X)$, then $\text{colorset}_u(Z)$ and then $\text{colorset}_u(Y)$. Also, the strategy would assign colors to v from $\text{colorset}_v(Y)$, then extraset_v and then $\text{colorset}_v(X)$ or $\text{colorset}_v(Z)$ depending whether $X < Z$ or otherwise. If the assigned colors of u and v are from different color sets, there is no confliction. Otherwise, there are two subcases to be considered:

(B-1) where u is assigned colors from $\text{colorset}_u(X)$ and v is assigned colors from $\text{colorset}_v(X)$ after exhausting all colors in $\text{colorset}_v(Y)$ and extraset_v. This means the weight w_v of v is very large. Since $w_u + w_v \leq \min\{3d_u, 3d_v\}$, and since u and v use $\text{colorset}(X) = \text{colorset}_u(X) \cup \text{colorset}_v(X)$ from opposite directions, u and v will not be assigned the same color;

(B-2) where u is assigned colors from $\text{colorset}_u(Y)$ and v is assigned colors from $\text{colorset}_v(Y)$. By similar analysis as in case (B-1), we can say that u and v will not be assigned the same color.

In Case C, the strategy would assign colors to u from $\text{colorset}_u(X)$, then extraset_u and then $\text{colorset}_u(Y)$ or $\text{colorset}_u(Z)$, and would assign colors to v from $\text{colorset}_v(Y)$, then extraset_v and then $\text{colorset}_v(X)$ or $\text{colorset}_v(Z)$. We can say that u and v cannot use colors from $\text{colorset}_u(Z)$ and $\text{colorset}_v(Z)$ at the same time. If it happens, the colors in $\text{colorset}_u(X)$, $\text{colorset}_v(Y)$, extraset_u and extraset_v are all used up, the total number of colors in these three color sets are no less than $3\min\{d_u, d_v\}$, which is a contradiction to the $c_u \leq 3d_u$ or $c_v \leq 3d_v$. So we have to consider three subcases:

(C-1) where u is assigned colors from $\text{colorset}_u(X)$ and v is assigned colors from $\text{colorset}_v(X)$ after exhausting all colors in $\text{colorset}_v(Y)$ and extraset_v. This means the weight w_v is very large. Since $w_u + w_v \leq \min\{3d_u, 3d_v\}$, and since u and v use $\text{colorset}(X) = \text{colorset}_u(X) \cup \text{colorset}_v(X)$ from opposite directions, u and v will not be assigned the same color;
(C-2) where u is assigned colors from extraset_u and v is assigned colors from extraset_v. This means all the colors in $\text{colorset}_u(X)$ and $\text{colorset}_v(Y)$ have been assigned to u and v, respectively. Since $w_u + w_v \leq \min\{3d_u, 3d_v\}$, and since u and v use $\text{extraset} = \text{extraset}_u \cup \text{extraset}_v$ from opposite directions (the parities of u and v are different), u and v will not be assigned the same color;
(C-3) where u is assigned colors from $\text{colorset}_u(Y)$ and v is assigned colors from $\text{colorset}_v(Y)$. By similar analysis as in case (C-1), we can say that u and v will not be assigned the same color.

For the whole triangle-free hexagonal graph, the maximal weight clique (2-clique) $c = \max_u\{c_u\}$ is a lower bound on the optimal value, and our algorithm uses at most $4\max_u\{\lceil c_u/3\rceil\}$ colors. Thus, the above strategy has an asymptotic approximation ratio of 4/3. □

From Lemma 1, we can easily have a 1-local asymptotic 4/3-competitive online algorithm for frequency allocation in triangle-free cellular networks.

4 Multicoloring in Hexagonal Graphs

In this section, we consider multicoloring hexagonal graphs. Our strategy works in two stages. In the first stage, each vertex assigns colors using local information on the weights of this vertex and its neighboring vertices. After the first stage, some vertices may be unsatisfied, i.e. not all of the necessary colors have been assigned, and the unsatisfied vertices, along with the edges connecting them, form a triangle-free graph. Applying the algorithm in the previous section, each vertex can be assigned colors, to satisfy all the remaining unsatisfied vertices, by using 1-local information. Combining these two stages, we have a 1-local algorithm for multicoloring hexagonal graphs.

We now describe the first stage, which is similar to the first stage in [12]. In [12], the algorithm needs to have the global information about the maximum weights of "all" 3-cliques in the graph (stage 1) so as to have an acyclic graph of the remaining unsatisfied vertices (for stage 2). As for an algorithm which is 1-local, only the maximal weights of the local 3-cliques will be available (stage 1) and a triangle-free hexagonal graph (which can be cyclic) will result (for stage 2). Consider vertex u with base color X. Let C_u be the maximal weight among the 3-cliques including u, and let $k_u = \lceil C_u/3 \rceil$. For the three base colors Red, Green and Blue, we define a cyclic order among them as Red $\rightarrow$ Green, Green $\rightarrow$ Blue and Blue $\rightarrow$ Red. If $X \rightarrow Y$, let m_u be the maximal weight of the neighboring vertices with color Y. We define color sets: $\text{colorset}_u(\text{Red}) = \{j \in \{1, \dots, 3k_u\} \mid j \equiv 1 \mod 3\}$, $\text{colorset}_u(\text{Green}) = \{j \in \{1, \dots, 3k_u\} \mid j \equiv 2 \mod 3\}$ and $\text{colorset}_u(\text{Blue}) = \{j \in \{1, \dots, 3k_u\} \mid j \equiv 0 \mod 3\}$. In the first stage, vertex u with base color X and weight w_u is assigned colors from these sets using the strategy described as follows:

1. Vertex u is assigned colors from $\text{colorset}_u(X)$ in bottom-to-top order.
2. If not enough and $m_u < k_u$, vertex u is assigned the upper $\min\{k_u - m_u, w_u - k_u\}$ colors from $\text{colorset}_u(Y)$.

After the first stage, each vertex has been assigned with some colors. The remaining graph contains only those vertices whose colors have not been totally satisfied, i.e., the number of assigned colors in vertex u is less than its weight.

Lemma 3 *The remaining graph is triangle-free*, *i.e.*, *contains no* 3-*clique*.

Proof If some vertex u is still unsatisfied, it must be that $w_u > \max\{k_u, 2k_u - m_u\}$. Thus, the remaining unsatisfied weight in vertex u is $w'_u = w_u - \max\{k_u, 2k_u - m_u\}$. For any three mutually-adjacent vertices (3-clique) u, v and t, since $\min\{C_u, C_v, C_t\} \geq w_u + w_v + w_t \geq 3\min\{w_u, w_v, w_t\}$, we have $\min\{k_u, k_v, k_t\} \geq \min\{w_u, w_v, w_t\}$. Without loss of generality, assume $w_u = \min\{w_u, w_v, w_t\}$, then obviously $k_u \geq w_u$ and $w'_u = 0$. From the description of coloring strategy, we can say that at most two of $\{w'_u, w'_v, w'_t\}$ are strictly positive, which means at least one of the vertices (u, v and t) has all its required colors totally assigned in the first stage. Therefore, the remaining graph contains no 3-clique, i.e., is a triangle-free hexagonal graph. □

Lemma 4 *The total weight of two neighboring vertices u and v in the remaining graph is at most* $\lfloor \min\{C_u, C_v\}/3 \rfloor$.

Proof For the remaining unsatisfied vertices, since $w'_u = w_u - \max\{k_u, 2k_u - m_u\}$, we have $w'_u \leq w_u - (2k_u - m_u) = w_u + m_u - 2k_u \leq C_u - 2k_u$. For any two adjacent unsatisfied vertices u and v, we can also get $w'_u + w'_v \leq w_u - k_u + w_v - k_v$. If $C_u \geq C_v$, which implies $k_u \geq k_v$, then we have $w'_u + w'_v \leq C_v - 2k_v \leq \lfloor C_v/3 \rfloor$ as $C_v \geq w_u + w_v$. Similarly, if $C_u \leq C_v$, which implies $k_u \leq k_v$, then we have $w'_u + w'_v \leq C_u - 2k_u \leq \lfloor C_u/3 \rfloor$. Thus, the total remaining weight of any two adjacent unsatisfied vertices is at most $\min\{\lfloor C_u/3 \rfloor, \lfloor C_v/3 \rfloor\}$. □

From Lemma 3, the remaining graph is triangle-free, so in the second stage, we can use the algorithm in Sect. 3 to process the remaining unsatisfied vertices. Each vertex gets the remaining weight information from its adjacent vertices and the total number of colors used in this stage is at most $4\max_u\{\lceil\lfloor C_u/3\rfloor/3\rceil\} = 4\max_u\lceil\frac{C_u-2}{9}\rceil$ (Theorem 2 and Lemma 4).

Combining these two stages, we use at most $\max_u(3k_u + 4\lceil\frac{C_u-2}{9}\rceil) \le \max_u(3\lceil\frac{C_u}{3}\rceil + 4\lceil\frac{C_u-2}{9}\rceil) \le \frac{13}{9}C_u + 7$ colors. Since C_u is a lower bound on the optimal solution, the asymptotic approximation ratio for our strategy is 13/9.

Thus, we have the following theorem.

Theorem 5 *There exists a* 1*-local asymptotic* 13/9*-competitive algorithm for the multicoloring problem in hexagonal graphs.*

5 Conclusion

We have given an asymptotic 13/9-approximation algorithm for multicoloring hexagonal graphs. This implies an asymptotic 13/9-competitive solution for the online frequency allocation problem, which involves servicing calls in each cell in a cellular network. The distributed algorithm is practical in the sense that frequency allocation can be done based on information about its neighbors and itself only. We note that, in fact, when calls are requested or released in a cell, a constant number of frequencies might have to be reassigned so as to actually achieve this asymptotic 13/9-competitive bound.

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